

LATTICE PATHS INSIDE A TABLE, I

D.YAQUIBI, , M. FARROKHI D. G., AND H. GHASEMIAN ZOERAM

ABSTRACT. A lattice path L in \mathbb{Z}^d of length k with steps in a given set $\mathbf{S} \subseteq \mathbb{Z}^d$, or \mathbf{S} -path for short, is a sequence $\nu_1, \nu_2, \dots, \nu_k \in \mathbb{Z}^d$ such that the steps $\nu_i - \nu_{i-1}$ lie in \mathbf{S} for all $i = 2, \dots, k$. Let $T_{m,n}$ be the $m \times n$ table in the first area of xy -axis and put $\mathbf{S} = \{(1, 0), (1, 1), (1, -1)\}$. Accordingly, let $\mathcal{I}_m(n)$ denote the number of \mathbf{S} -paths starting from the first column and ending at the last column of T . We will study the numbers $\mathcal{I}_m(n)$ and give explicit formulas for special values of m and n . As a result, we prove a conjecture of *Alexander R. Povolotsky*. We conclude the paper with some applications to Fibonacci and Pell-Lucas numbers and posing an open problem.

1. INTRODUCTION

A *lattice path* L in \mathbb{Z}^d is a path in the d -dimensional integer lattice \mathbb{Z}^d , which uses only points of the lattice; that is a sequence $\nu_1, \nu_2, \dots, \nu_k$, where $\nu_i \in \mathbb{Z}^d$ for all i (see [9, 10]). The vectors $\nu_2 - \nu_1, \nu_3 - \nu_2, \dots, \nu_k - \nu_{k-1}$ are called the *steps* of L . Recall that a *Dyck path* is a lattice path in \mathbb{Z}^2 starting from $(0, 0)$ and ending at a point $(2n, 0)$ (for some $n \geq 0$) consisting of up-steps $(1, 1)$ and down-steps $(1, -1)$, which never passes below the x -axis. It is well known that Dyck paths of length $2n$ are counted by the n^{th} -*Catalan number* $\mathcal{C}_n = \frac{1}{n+1} \binom{2n}{n}$. The Catalan numbers arise in many combinatorial problems, see Stanley [12] for an extensive study of these numbers.

Let $T_{m,n}$ be the $m \times n$ table in the first quadrant composed of mn unit squares, whose (x, y) -blank is located in the x^{th} -column from the left and the y^{th} -row from the bottom hand side of $T_{m,n}$. For a set $\mathbf{S} \subseteq \mathbb{Z}^d$ of steps, let $L(i, j; s, t : \mathbf{S})$ denote the set of all lattice paths in $T_{m,n}$ starting form (i, j) -blank and ending at (s, t) -blank with steps in \mathbf{S} , where $1 \leq i, s \leq m$ and $1 \leq j, t \leq n$. The number of such lattice paths is denoted by $l(i, j; s, t : \mathbf{S})$. For example, assuming $\mathbf{S} := \{(1, 0), (1, 1), (1, -1)\}$, the set of all lattice paths in the table T starting

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from $(1, 1)$ and ending at $(n, 1)$, where allowed to move only to the right (up, down or straight) is shown by $L(1, 1; n, 1 : \mathbf{S})$ and the number of such lattice paths, namely $l(1, 1; n, 1 : \mathbf{S})$, is the n^{th} -Motzkin number.

Lattice paths starting from the first column and ending at the n column of $T_{m,n}$ with steps in \mathbf{S} , \mathbf{S} being as above, are called *perfect lattice paths*, and the number of all perfect lattice paths is denoted by $\mathcal{I}_m(n)$. Indeed,

$$\mathcal{I}_m(n) = \sum_{i,j=1}^m l(1, i; n, j; \mathbf{S}).$$

Figure 1 shows the number of all perfect lattice paths for $m = 2$ and $n = 3$. Clearly, $l(1, i; n, j : \mathbf{S}) = l(1, i'; n, j' : \mathbf{S})$ when $i + i' = m + 1$ and $j + j' = m + 1$.

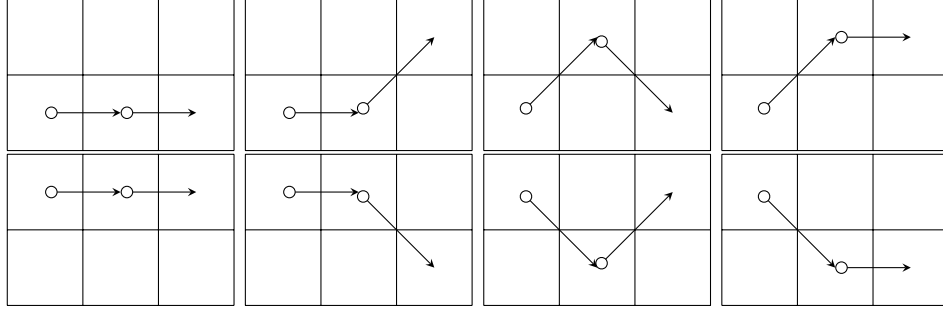


FIGURE 1. All perfect lattice paths in $T_{2,3}$.

We intend to evaluate $\mathcal{I}_m(n)$ for special cases of (m, n) . In section 2, we obtain $\mathcal{I}_m(n)$ when $m \geq n$. Also, we prove a conjecture of *Alexander R. Povolotsky* posed in [8]. In section 3, we shall compute $\mathcal{I}_m(n)$ for small values of m , namely $m = 1, 2, 3, 4$. Finally, we present some results for $\mathcal{I}_5(n)$ and use Fibonacci and Pell-Lucas numbers to prove some relations concerning perfect lattice paths.

2. $\mathcal{I}_n(n)$ VS ALEXANDER R. POVOLOTSKY'S CONJECTURE

Let $T = T_{m,n}$ be the $m \times n$ table. For positive integers $1 \leq i, t \leq m$ and $1 \leq s \leq n$, the number of all perfect lattice paths from $(1, i)$ to (s, t) in T is denoted by $\mathcal{D}^i(s, t)$, that is, $\mathcal{D}^i(s, t) = l(1, i; s, t : \mathbf{S})$. Also, we put

$$\mathcal{D}_{m,n}(s, t) = \sum_{i=1}^m \mathcal{D}^i(s, t).$$

In case we are working in a single table, say T as above, we may simply use $\mathcal{D}(s, t)$ for $\mathcal{D}_{m,n}(s, t)$. Also, we put $\mathcal{D}_n(s, t) := \mathcal{D}_{n,n}(s, t)$. Clearly,

$\mathcal{D}(s, t)$ is the number of all perfect lattice paths from first column to the (s, t) -blank of T . The values of $\mathcal{D}(s, n)$ is computed in [4] in the cases where $m = n$ and T is a square table. By symmetry, $\mathcal{D}(s, t) = \mathcal{D}(s, t')$ when $t + t' = m + 1$. Table 1 illustrates the values of $\mathcal{D}(6, t)$, for all $1 \leq t \leq 6$, where the number in (s, t) -blank of T determines the number $\mathcal{D}(s, t)$.

					$\mathcal{D}(6, t)$
1	2	5	13	35	96
1	3	8	22	61	170
1	3	9	26	74	209
1	3	9	26	74	209
1	3	8	22	61	170
1	2	5	13	35	96

TABLE 1. Values of $\mathcal{D}(6, t)$

Theorem 2.1. *For any positive integer n we have*

$$\mathcal{I}_n(n) = 3\mathcal{I}_{n-1}(n-1) + 3^{n-1} - 2\mathcal{D}_{n-1}(n-1, n-1).$$

Proof. Let $T := T_{n,n}$ and $T' := T_{n-1,n-1}$ with T' in the left-bottom side of T . Clearly, the number of perfect lattice paths of T which never meet the n^{th} row of T is

$$\mathcal{I}_{n-1}(n) = 3\mathcal{I}_{n-1}(n-1) - 2\mathcal{D}_{n-1}(n-1, n-1).$$

To obtain the number of all perfect lattice paths we must count those who meet the n^{th} -row of T , that is equal to 3^{n-1} . Thus $\mathcal{I}_n(n) - \mathcal{I}_{n-1}(n) = 3^{n-1}$, from which the result follows. \square

Michael Somos [4] gives the following recurrence relation for $\mathcal{D}(n, n)$.

Theorem 2.2. *Inside the square $n \times n$ table we have*

$$n\mathcal{D}(n, n) = 2n\mathcal{D}(n-1, n-1) + 3(n-2)\mathcal{D}(n-2, n-2).$$

Utilizing Theorems 2.1 and 2.2 for $\mathcal{D}_n(n, n)$, we can prove a conjecture of Alexander R. Povolotsky posed in [8] as follows:

Conjecture 2.3. The following identity holds for the numbers $\mathcal{I}_n(n)$.

$$\begin{aligned} (n+3)\mathcal{I}_{n+4}(n+4) &= 27n\mathcal{I}_n(n) + 27\mathcal{I}_{n+1}(n+1) \\ &\quad - 9(2n+5)\mathcal{I}_{n+2}(n+2) + (8n+2)\mathcal{I}_{n+3}(n+3). \end{aligned}$$

Proof. Put

$$\begin{aligned} A &= (n+3)\mathcal{I}_{n+4}(n+4), \\ B &= (8n+21)\mathcal{I}_{n+3}(n+3), \\ C &= 9(2n+5)\mathcal{I}_{n+2}(n+2), \\ D &= 27\mathcal{I}_{n+1}(n+1), \\ E &= 27n\mathcal{I}_n(n). \end{aligned}$$

Using Theorem 2.1, we can write

$$\begin{aligned} A &= (3n+9)\mathcal{I}_{n+3}(n+3) + (n+3)3^{n+3} - (2n+6)\mathcal{D}(n+3, n+3) \\ &= (8n+21)\mathcal{I}_{n+3}(n+3) - (5n+12)\mathcal{I}_{n+3}(n+3) + (n+3)3^{n+3} \\ &\quad - (2n+6)\mathcal{D}(n+3, n+3) \\ &= B + (n+3)3^{n+3} - (5n+12)\mathcal{I}_{n+3}(n+3) \\ &\quad - (2n+6)\mathcal{D}(n+3, n+3). \end{aligned} \tag{2.1}$$

Utilizing Theorem 2.1 once more for $\mathcal{I}_{n+3}(n+3)$ and $\mathcal{I}_{n+2}(n+2)$ yields

$$\begin{aligned} A &= B + (n+3)3^{n+3} - (5n+12)3^{n+2} \\ &\quad - (18n+45)\mathcal{I}_{n+2}(n+2) - (2n+6)\mathcal{D}(n+3, n+3) \\ &\quad + (10n+24)\mathcal{D}(n+2, n+2) + (3n+9)\mathcal{I}_{n+2}(n+2) + (n+3)3^{n+3} \\ &= B - C - (5n+12)3^{n+2} - (2n+6)\mathcal{D}(n+3, n+3) \\ &\quad + (10n+24)\mathcal{D}(n+2, n+2) + 9n\mathcal{I}_{n+1}(n+1) \\ &\quad + 27\mathcal{I}_{n+1}(n+1) + (3n+9)3^{n+1} - (6n+18)\mathcal{D}(n+1, n+1). \end{aligned}$$

It can be easily shown that

$$\begin{aligned} A &= B - C + D \\ &\quad + (n+3)3^{n+3} - (2n+6)\mathcal{D}(n+3, n+3) - (5n+12)3^{n+2} \\ &\quad + (10n+24)\mathcal{D}(n+2, n+2) + 9n\mathcal{I}_{n+1}(n+1) \\ &\quad + (3n+9)3^{n+1} - (6n+18)\mathcal{D}(n+1, n+1). \end{aligned} \tag{2.2}$$

Replacing $9n\mathcal{I}_{n+1}(n+1)$ by $27n\mathcal{I}_n(n) + n3^{n+2} - 18n\mathcal{I}_n(n)$ in 2.2 gives

$$\begin{aligned} A &= B - C + D + E \\ &\quad - (2n+6)\mathcal{D}(n+3, n+3) + (10n+24)\mathcal{D}(n+2, n+2) \\ &\quad - 18n\mathcal{D}(n, n) - (6n+18)\mathcal{D}(n+1, n+1). \end{aligned}$$

Since the coefficient of $\mathcal{D}(n+3, n+3)$ is $2(n+3)$, it follow from Theorem 2.2 that

$$\begin{aligned}
A &= B - C + D + E - (4n + 12)\mathcal{D}(n + 2, n + 2) - 18n\mathcal{D}(n, n) \\
&\quad + (10n + 24)\mathcal{D}(n + 2, n + 2) - (6n + 6)\mathcal{D}(n + 1, n + 1) \\
&\quad - (6n + 18)\mathcal{D}(n + 1, n + 1) \\
&= B - C + D + E - (4n + 12)\mathcal{D}(n + 2, n + 2) \\
&\quad - (6n + 6)\mathcal{D}(n + 1, n + 1) + 18n\mathcal{D}(n, n) - 18n\mathcal{D}(n, n) \\
&\quad - (12n + 24)\mathcal{D}(n + 1, n + 1) + (6n + 18)\mathcal{D}(n + 1, n + 1) \\
&= B - C + D + E,
\end{aligned}$$

as required. \square

Theorem 2.4. *Inside the $m \times n$ table we have*

$$\mathcal{I}_m(n) = m3^{n-1} - 2 \sum_{s=1}^{n-1} 3^{n-s-1} \mathcal{D}(s, 1). \quad (2.3)$$

Proof. Let $T := T_{m,n}$. The number of all lattice paths from the first column to the last column is simply $n3^{n-1}$ if they are allowed to get out of T . Now we count all lattice paths that go out of T in some step. First observe that the number of lattice paths that leave T from the bottom row equals to those leave T from the the top row in the first times. Suppose a lattice path goes out of T from the bottom in column s for the first times. The number of all partial lattice paths from the first column to the $(s-1, 1)$ -blank is simply $\mathcal{D}(s-1, 1)$, and every such path continues in 3^{n-s} ways until it reaches the last column of T . Hence we have $3^{n-s}\mathcal{D}(s-1, 1)$ paths leave the table T from the bottom in column s for any $s = 2, \dots, n$. Hence, the number of perfect lattice paths is simply

$$\begin{aligned}
\mathcal{I}_m(n) &= m3^{n-1} - 2 \sum_{i=2}^n 3^{n-s} \mathcal{D}(s-1, 1) \\
&= m3^{n-1} - 2 \sum_{s=1}^{n-1} 3^{n-s-1} \mathcal{D}(s, 1),
\end{aligned}$$

as required. \square

Example 2.5. Let T be the square 6×6 table. In Table 1, every blank represents the number of all perfect lattice paths from first column to that blank. Summing up the last column yields

$$\mathcal{I}_6(6) = 96 + 170 + 209 + 209 + 170 + 96 = 950.$$

Now, utilizing Theorem 2.4, we calculate $\mathcal{I}_6(6)$ in another way, as follows:

$$\begin{aligned}\mathcal{I}_6(6) &= 6 \cdot 3^{6-1} - 2(3^{6-1-1}\mathcal{D}(1, 1) + 3^{6-2-1}\mathcal{D}(2, 1) + 3^{6-3-1}\mathcal{D}(3, 1) \\ &\quad + 3^{6-4-1}\mathcal{D}(4, 1) + 3^{6-5-1}\mathcal{D}(5, 1)) \\ &= 1458 - 2(3^4 \cdot 1 + 3^3 \cdot 2 + 3^2 \cdot 5 + 3^1 \cdot 13 + 3^0 \cdot 35) = 950.\end{aligned}$$

Remind that the number of lattice paths $L(1, 1; n+1, 1 : \mathbf{S})$ in \mathbb{Z}^2 that never slides below the x -axis, is the n^{th} -Motzkin number ($n \geq 0$), denoted by \mathcal{M}_n . Motzkin numbers begin with 1, 1, 2, 4, 9, 21, ... (see [2]) and can be expressed in terms of binomial coefficients and Catalan numbers via

$$\mathcal{M}_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \mathcal{C}_k.$$

Trinomial triangles are defined by the same steps $(1, 1)$, $(1, -1)$ and $(1, 0)$ (in our notation) with no restriction by starting from a fixed blank. The number of ways to reach a blank is simply the sum of three numbers in the adjacent previous column. The k^{th} -entry of the n^{th} column is denoted by $\binom{n}{k}_2$, where columns start by 0. The middle entries of the Trinomial triangle, namely 1, 1, 3, 7, 19, ... (see [6]) are studied by Euler. Analogously, Motzkin triangle are defined by recurrence sequence

$$\mathcal{T}(n, k) = \mathcal{T}(n-1, k-2) + \mathcal{T}(n-1, k-1) + \mathcal{T}(n-1, k),$$

for all $1 \leq k \leq n-1$ and satisfy

$$\mathcal{T}(n, n) = \mathcal{T}(n-1, n-2) + \mathcal{T}(n-1, n-1)$$

for all $n \geq 1$ (see [5]).

Table 2 illustrates initial parts of the above triangles with Motzkin triangle in the left and trinomial triangle in the right. For a positive integer $1 \leq s \leq n$, each entry of the column $\mathcal{D}_s(s, 1)$ is the sum of all entries in the s^{th} -row in the rotated Motzkin triangle, that is, $\mathcal{D}_s(s, 1) = \sum_{i=1}^s \mathcal{T}(s, i)$. For example,

$$\mathcal{D}(4, 1) = \mathcal{T}(4, 1) + \mathcal{T}(4, 2) + \mathcal{T}(4, 3) + \mathcal{T}(4, 4) = 4 + 5 + 3 + 1 = 13.$$

The entries in the first column of rotated Motzkin triangle are indeed the Motzkin numbers.

Lemma 2.6. *Inside the square $n \times n$ table we have*

$$\mathcal{D}(s, 1) = 3\mathcal{D}(s-1, 1) - \mathcal{M}_{s-2},$$

for all $1 \leq s \leq n$.

$\mathcal{D}_s(s, 1)$												
1	1									1		
2	1	1								1	1	
5	2	2	1							1	2	1
13	4	5	3	1						1	3	3
35	9	12	9	4	1				1	3	3	1
96	21	30	25	14	5	1		1	4	6	4	1

TABLE 2. Motzkin triangle (left) and trinomial triangle (right) rotates 90° clockwise

Proof. Let $T := T_{n,n}$. By the definition, $\mathcal{D}(s, 1)$ is the number of all lattice paths from the first column to $(s, 1)$ -blank. This number equals the number of lattice paths from $(s, 1)$ -blank to the first column with reverse steps that lie inside the table T , which is equal to 3^{s-1} minus those paths that leave T at some point. Consider all those lattice paths starting from $(s, 1)$ -blank with reverse steps that leaves T at $(i, 0)$ for the first time, where $1 \leq i \leq s-1$. Clearly, the number of such paths are $3^{i-1}\mathcal{M}_{s-i-1}$. Thus

$$\mathcal{D}(s, 1) = 3^{s-1} - \sum_{i=1}^{s-1} 3^{i-1}\mathcal{M}_{s-i-1},$$

from which it follows that $\mathcal{D}(s, 1) = 3\mathcal{D}(s-1, 1) - \mathcal{M}_{s-2}$, as required. \square

Example 2.7. Consider the Table 2. Using Lemma 2.6 we can calculate $\mathcal{D}(6, 1)$ as

$$\mathcal{D}(6, 1) = 3\mathcal{D}(5, 1) - \mathcal{M}_4 = 3 \cdot 35 - 9 = 96.$$

Corollary 2.8. *Inside the $n \times n$ table we have*

$$\mathcal{I}_n(n) = (n+2)3^{n-2} + 2 \sum_{k=0}^{n-3} (n-k-2)3^{n-k-3}\mathcal{M}_k.$$

Proof. The result follows from Theorem 2.4 and Lemma 2.6. \square

The next result shows that the number of perfect lattice paths in $T_{m,n}$ is independent of the number m or rows provided that m is big enough.

Theorem 2.9. *Inside the $m \times n$ table ($m \geq n$) we have*

$$\mathcal{I}_{m+1}(n) - \mathcal{I}_m(n) = \sum_{i=0}^{n-1} \mathcal{D}(i, 1) \mathcal{D}(n-i, 1).$$

Proof. Consider the table $T := T_{m,n}$. We construct the table T' by adding a new row $m+1$ at the top of T . Now to count the number of all perfect lattice paths in T' , it is sufficient to consider perfect lattice paths that reach to the new row $m+1$ for the first time. Assume a perfect lattice path reach to row $m+1$ at column i for the first time. Then its initial part from column 1 to column $i-1$ is a lattice path from the first column of T to $(i-1, m)$ -blank. Also, its terminal part from column i to column n is a lattice path from $(i, m+1)$ -blank of T' to its last column, which is in one to one correspondence with a lattice path from (i, m) -blank of T to its last column as $m \geq n$. Hence, the number of such paths is simply $\mathcal{D}(i-1, m) \mathcal{D}(n-i+1, m)$, which is equal to $\mathcal{D}(i-1, 1) \mathcal{D}(n-i+1, 1)$ by symmetry. Therefore

$$\mathcal{I}_{m+1}(n) - \mathcal{I}_m(n) = \sum_{i=1}^n \mathcal{D}(i-1, 1) \mathcal{D}(n-i+1, 1)$$

and the result follows. \square

Corollary 2.10. *For $m \geq n$ we have*

$$\begin{aligned} \mathcal{I}_m(n) &= (n+2)3^{n-2} + (m-n) \sum_{i=0}^{n-1} \mathcal{D}(i, 1) \mathcal{D}(n-i, 1) \\ &\quad + 2 \sum_{k=0}^{n-3} (n-k-2) 3^{n-k-3} \mathcal{M}_k. \end{aligned}$$

Proof. Let $m = n + k$, where k is a positive integer. Then

$$\begin{aligned} \mathcal{I}_m(n) - \mathcal{I}_n(n) &= (\mathcal{I}_m(n) - \mathcal{I}_{m-1}(n)) + \cdots + (\mathcal{I}_{n+1}(m) - \mathcal{I}_n(m)) \\ &= (m-n) \sum_{i=0}^{n-1} \mathcal{D}(i, 1) \mathcal{D}(n-i, 1). \end{aligned}$$

Now the result follows from Corollary 2.8. \square

Theorem 2.11. *Inside the $m \times n$ table with $m \geq 2n-2$ we have*

- (i) $\sum_{i=0}^{n-1} \mathcal{D}(i, n) \mathcal{D}(n-i, n) = 3^{n-1}$;
- (ii) $\sum_{i=1}^{n-1} \mathcal{D}(i, n) \mathcal{D}(n-i, n) = \sum_{i=0}^{n-2} 3^{n-i-1} \mathcal{M}_i$;
- (iii) $\mathcal{I}_m(n) = (3m-2n+2)3^{n-2} + 2 \sum_{k=0}^{n-3} (n-k-2) 3^{n-k-3} \mathcal{M}_k$.

Proof. (i) Let $T := T_{m,n}$ with $m = 2n - 2$ and T' be the table obtained by adding a new row in the middle of T . It is sufficient to obtain $\mathcal{I}_{m+1}(n) - \mathcal{I}_m(n)$. Clearly, the number of perfect lattice paths reaching to any (i, n) -blank of T or T' is the same for all $i = 1, \dots, n - 1$. On the other hand, the number of all perfect lattice paths of T' reaching at (n, n) -blank is 3^{n-1} since we may begin the paths from the last (n, n) -blank and apply reverse steps with no limitation until to reach the first column. Thus

$$3^{n-1} = \mathcal{I}_{m+1}(n) - \mathcal{I}_m(n) = \sum_{i=0}^{n-1} \mathcal{D}(i, 1) \mathcal{D}(n - i, 1).$$

(ii) Put $\mathcal{D}(0, 1) = 1$. Then

$$\mathcal{D}(n, 1) = 3^{n-1} - \sum_{i=1}^{n-1} \mathcal{D}(i, 1) \mathcal{D}(n - i, 1).$$

On the other hand, by Lemma 2.6, we have

$$\mathcal{D}(n, 1) = 3^{n-1} - \sum_{i=0}^{n-2} 3^{n-i-2} \mathcal{M}_i,$$

from which the result follows.

(iii) It follows from (i) and Corollary 2.10. □

Lemma 2.12. *Inside the $n \times n$ table we have*

$$\mathcal{D}(n, k + 2) - \mathcal{D}(n, k) = \sum_{i=1}^{n-1} (\mathcal{D}(i, k + 3) - \mathcal{D}(i, k - 1))$$

for all $1 \leq k \leq n$.

Proof. For $n = 2$, the result is trivially true. For any $\ell < n$ we have

$$\begin{aligned} \mathcal{D}(\ell + 1, k + 2) &= \mathcal{D}(\ell, k + 3) + \mathcal{D}(\ell, k + 2) + \mathcal{D}(\ell, k + 1) \\ \mathcal{D}(\ell + 1, k) &= \mathcal{D}(\ell, k + 1) + \mathcal{D}(\ell, k) + \mathcal{D}(\ell, k - 1), \end{aligned}$$

which imply that

$$\begin{aligned} &\mathcal{D}(\ell + 1, k + 2) - \mathcal{D}(\ell + 1, k) \\ &= \mathcal{D}(\ell, k + 3) - \mathcal{D}(\ell, k - 1) + (\mathcal{D}(\ell, k + 2) - \mathcal{D}(\ell, k)). \end{aligned}$$

Thus

$$\mathcal{D}(n, k + 2) - \mathcal{D}(n, k) = \sum_{i=1}^{n-1} (\mathcal{D}(i, k + 3) - \mathcal{D}(i, k - 1))$$

as $\mathcal{D}(1, k + 2) - \mathcal{D}(1, k) = 0$. This completed the proof. □

3. CONCLUDING SOME REMARKS AND ONE CONJECTURE

In this section, we shall compute $\mathcal{I}_m(n)$ for $m = 1, 2, 3, 4$ and arbitrary positive integers n . Some values of the $\mathcal{I}_3(n)$ and $\mathcal{I}_4(n)$ are given in [3] and [7], respectively.

Lemma 3.1. $\mathcal{I}_1(n) = 1$ and $\mathcal{I}_2(n) = 2^n$ for all $n \geq 1$.

Let x and y be arbitrary real numbers. By the binomial theorem, we have the following identity,

$$x^n + y^n = (x + y)^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x + y)^{n-2k},$$

where $n \geq 1$. This identity also can be rewritten as

$$x^n + y^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x + y)^{n-2k}, \quad (3.1)$$

where $\binom{r}{-1} = 0$. Pell-Lucas sequence [11] is defined as $\mathcal{Q}_1 = 1$, $\mathcal{Q}_2 = 3$, and $\mathcal{Q}_n = 2\mathcal{Q}_{n-1} + \mathcal{Q}_{n-2}$ for all $n \geq 3$. It can also be defined by the so called *Binet formula* as $\mathcal{Q}_n = (\alpha^n + \beta^n)/2$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ are solutions of the quadratic equation $x^2 = 2x + 1$.

Lemma 3.2. For all $n \geq 1$ we have $\mathcal{I}_3(n) = \mathcal{Q}_{n+1}$.

Proof. The number of lattice paths to entries in columns $n-2$, $n-1$ and n of $T_{3,n}$ looks like

$n-2$	$n-1$	n
x	$x+y$	$3x+2y$
y	$2x+y$	$4x+3y$
x	$x+y$	$3x+2y$

which imply that $\mathcal{I}_3(n-2) = 2x+y$, $\mathcal{I}_3(n-1) = 4x+3y$ and $\mathcal{I}_3(n) = 10x+7y$. Thus the following linear recurrence exists for \mathcal{I}_3 .

$$\mathcal{I}_3(n) = 2\mathcal{I}_3(n-1) + \mathcal{I}_3(n-2). \quad (3.2)$$

Since $\mathcal{I}_3(1) = \mathcal{Q}_2 = 3$ and $\mathcal{I}_3(2) = \mathcal{Q}_3 = 7$, it follows that $\mathcal{I}_3(n) = \mathcal{Q}_{n+1}$ for all $n \geq 1$, as required. \square

Corollary 3.3. *Let n be a positive integer. Then*

$$\mathcal{I}_3(n) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \left[\binom{n-k+1}{k} + \binom{n-k}{k-1} \right] 2^{n-2k}.$$

Proof. It is sufficient to put $x = \alpha$ and $y = \beta$ in 3.1. \square

The Fibonacci sequence [1] starts with the integers 0 and 1, and every other term is the sum of the two preceding ones, that is, $\mathcal{F}_0 = 0$, $\mathcal{F}_1 = 1$, and $\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}$ for all $n \geq 2$. This recursion gives the Binet's formula $\mathcal{F}_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Lemma 3.4. *For all $n \geq 1$ we have $\mathcal{I}_4(n) = 2\mathcal{F}_{2n+1}$.*

Proof. The number of lattice paths to entries in columns $n-2$, $n-1$ and n of $T_{4,n}$ looks like

$n-2$	$n-1$	n
x	$x+y$	$2x+3y$
y	$x+2y$	$3x+5y$
y	$x+2y$	$3x+5y$
x	$x+y$	$2x+3y$

which imply that $\mathcal{I}_4(n-2) = 2x+2y$, $\mathcal{I}_4(n-1) = 4x+6y$ and $\mathcal{I}_4(n) = 10x+16y$. Hence we get the following linear recurrence for \mathcal{I}_4 .

$$\mathcal{I}_4(n) = 3\mathcal{I}_4(n-1) - \mathcal{I}_4(n-2). \quad (3.3)$$

On the other hand,

$$\begin{aligned} \mathcal{F}_{2n+1} &= \mathcal{F}_{2n} + \mathcal{F}_{2n-1} \\ &= 2\mathcal{F}_{2n-1} + \mathcal{F}_{2n-2} \\ &= 3\mathcal{F}_{2n-1} - \mathcal{F}_{2n-3} \\ &= 3\mathcal{F}_{2(n-1)+1} - \mathcal{F}_{2(n-2)+1}. \end{aligned}$$

Now since $\mathcal{I}_4(1) = 2\mathcal{F}_3$ and $\mathcal{I}_4(2) = 2\mathcal{F}_5$, it follows that $\mathcal{I}_4(n) = 2\mathcal{F}_{2n+1}$ for all $n \geq 1$. The proof is complete. \square

Corollary 3.5. *For all $n \geq 1$ we have*

$$\mathcal{I}_4(n) = \sum_{k=0}^n (-1)^k \left[\frac{2n+1}{k} \binom{2n-k}{k-1} \right] 5^{n-k}. \quad (3.4)$$

Proof. It is sufficient to put $x = \varphi$ and $y = \psi$ in 3.1. \square

$n - 3$	$n - 2$	$n - 1$	n
x_1	$x_1 + x_2$	$2x_1 + 2x_2 + x_3$	$4x_1 + 6x_2 + 3x_3$
x_2	$x_1 + x_2 + x_3$	$2x_1 + 4x_2 + 2x_3$	$6x_1 + 10x_2 + 6x_3$
x_3	$2x_2 + x_3$	$2x_1 + 4x_2 + 3x_3$	$6x_1 + 12x_2 + 7x_3$
x_2	$x_1 + x_2 + x_3$	$2x_1 + 4x_2 + 2x_3$	$6x_1 + 10x_2 + 6x_3$
x_1	$x_1 + x_2$	$2x_1 + 2x_2 + x_3$	$4x_1 + 6x_2 + 3x_3$

from which it follows that

$$\begin{aligned}
\mathcal{I}_5(n - 3) &= 2x_1 + 2x_2 + x_3, \\
\mathcal{I}_5(n - 2) &= 4x_1 + 6x_2 + 3x_3, \\
\mathcal{I}_5(n - 1) &= 10x_1 + 16x_2 + 9x_3, \\
\mathcal{I}_5(n) &= 28x_1 + 44x_2 + 25x_3
\end{aligned}$$

Clearly,

$$\mathcal{I}_5(n) = \ell_1 \mathcal{I}_5(n - 1) + \ell_2 \mathcal{I}_5(n - 2) + \ell_3 \mathcal{I}_5(n - 3)$$

for some ℓ_1, ℓ_2, ℓ_3 , and that the coefficient matrix of the table T is

$$\mathcal{C}(T) = \begin{bmatrix} 10 & 4 & 2 \\ 16 & 6 & 2 \\ 9 & 3 & 1 \end{bmatrix}. \text{ It is obvious that } \det(\mathcal{C}(T)) = -2^{\lfloor \frac{5}{2} \rfloor} = -4.$$

4. SOME RESULT ABOUT PERFECT LATTICE PATHS BY USING OF FIBONACCI AND PELL-LUCAS NUMBERS

In this section, we obtain some results about perfect lattice paths in the $5 \times n$ table. Also, we get some relations and properties about Fibonacci and Pell-Lucas sequences by the aid of perfect lattice paths.

Proposition 4.1. *Inside the $5 \times n$ table we have*

$$\mathcal{D}(s + 2, 1) = \mathcal{I}_5(s) \quad \text{and} \quad \mathcal{D}(s + 2, 3) = 2\mathcal{I}_5(s) - 1$$

for all $1 \leq s \leq n$.

Proof. From the table in Example 3.7, it follows simply that $\mathcal{I}_5(s) = \mathcal{D}(s + 2, 1)$ for all $s \geq 1$. Also, from the table, it follows that

$$2\mathcal{D}(s + 1, 1) - \mathcal{D}(s + 1, 3) = 2\mathcal{D}(s, 1) - \mathcal{D}(s, 3)$$

for all $s \geq 1$, that is, $2\mathcal{D}(s, 1) - \mathcal{D}(s, 3)$ is constant. Since $2\mathcal{D}(1, 1) - \mathcal{D}(1, 3) = 1$, we get $2\mathcal{D}(s + 2, 1) - \mathcal{D}(s + 2, 3) = 1$, from which the result follows. \square

Proposition 4.2. *Inside the $5 \times n$ table we have*

$$\mathcal{D}(s, 1) \times \mathcal{D}(s + t, 3) - \mathcal{D}(s, 3) \times \mathcal{D}(s + t, 1) = \sum_{i=s}^{s+t-1} \mathcal{D}(i, 2)$$

for all $1 \leq s, t \leq n$.

Proof. From Proposition 4.1, we know that $\mathcal{D}(s, 3) = 2\mathcal{D}(s, 1) - 1$ for all $1 \leq s \leq n$. Then

$$\begin{aligned} & \mathcal{D}(s, 1)\mathcal{D}(s+t, 3) - \mathcal{D}(s, 3)\mathcal{D}(s+t, 1) \\ &= \mathcal{D}(s, 1)(2\mathcal{D}(s+t, 1) - 1) - (2\mathcal{D}(s, 1) - 1)\mathcal{D}(s+t, 1) \\ &= 2\mathcal{D}(s, 1)\mathcal{D}(s+t, 1) - \mathcal{D}(s, 1) - 2\mathcal{D}(s, 1)\mathcal{D}(s+t, 1) + \mathcal{D}(s+t, 1) \\ &= \mathcal{D}(s+t, 1) - \mathcal{D}(s, 1). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathcal{D}(s+t, 1) - \mathcal{D}(s, 1) \\ &= \mathcal{D}(s+t-1, 1) + \mathcal{D}(s+t-1, 2) - \mathcal{D}(s, 1) \\ &= \mathcal{D}(s+t-2, 1) + \mathcal{D}(s+t-2, 2) + \mathcal{D}(s+t-1, 2) - \mathcal{D}(s, 1) \\ & \vdots \\ &= \sum_{i=s}^{s+t-1} \mathcal{D}(i, 2) + \mathcal{D}(s, 1) - \mathcal{D}(s, 1) \\ &= \sum_{i=s}^{s+t-1} \mathcal{D}(i, 2), \end{aligned}$$

from which the result follows. \square

Proposition 4.3. *Inside the $4 \times n$ table we have*

$$\mathcal{D}(s, 1) = \mathcal{F}_{2s-1} \quad \text{and} \quad \mathcal{D}(s, 2) = \mathcal{F}_{2s}$$

for all $s \geq 1$. As a result,

$$\mathcal{D}(s, 1) \times \mathcal{D}(s+t, 2) - \mathcal{D}(s, 2) \times \mathcal{D}(s+t, 1) = \mathcal{D}(s, 2).$$

for all $s, t \geq 1$.

Proof. Clearly $\mathcal{D}(1, 1) = \mathcal{D}(1, 2) = \mathcal{F}_1 = \mathcal{F}_2 = 1$. Now since

$$\begin{aligned} \mathcal{D}(s, 1) &= \mathcal{D}(s-1, 1) + \mathcal{D}(s-1, 2), \\ \mathcal{D}(s, 2) &= 2\mathcal{D}(s-1, 2) + \mathcal{D}(s-1, 1). \end{aligned}$$

we may prove, by using induction that, $\mathcal{D}(s, 1) = \mathcal{F}_{2s-1}$ and $\mathcal{D}(s, 2) = \mathcal{F}_{2s}$ for all $s \geq 1$. The second claim follows from the fact that

$$\mathcal{F}_{2s-1}\mathcal{F}_{2s+2t} - \mathcal{F}_{2s}\mathcal{F}_{2s+2t-1} = \mathcal{F}_{2s}.$$

The proof is complete. \square

Proposition 4.4. *Inside the $4 \times n$ table we have*

$$\mathcal{I}_4(2s+1) = \frac{1}{4}\mathcal{I}_4(s+1)^2 + \mathcal{D}(s, 2)^2$$

for all $1 \leq s \leq n$.

Proof. Following Lemma 3.4 and Proposition 4.3, it is enough to show that

$$2\mathcal{F}_{4s+3} = \mathcal{F}_{2s+3}^2 + \mathcal{F}_{2s}^2.$$

First observe that the equation $\mathcal{F}_{2n-1} = \mathcal{F}_n^2 + \mathcal{F}_{n-1}^2$ yields $\mathcal{F}_{4s+1} = \mathcal{F}_{2s+1}^2 + \mathcal{F}_{2s+2}^2$ and $\mathcal{F}_{4s+5} = \mathcal{F}_{2s+3}^2 + \mathcal{F}_{2s+2}^2$. Now, by combining these two formulas, we obtain

$$\begin{aligned} \mathcal{F}_{2s+3}^2 + \mathcal{F}_{2s}^2 &= \mathcal{F}_{4s+5} + \mathcal{F}_{4s+1} - (\mathcal{F}_{2s+1}^2 + \mathcal{F}_{2s+2}^2) \\ &= \mathcal{F}_{4s+4} + \mathcal{F}_{4s+3} + \mathcal{F}_{4s+1} - \mathcal{F}_{4s+3} \\ &= \mathcal{F}_{4s+3} + \mathcal{F}_{4s+2} + \mathcal{F}_{4s+1} \\ &= 2\mathcal{F}_{4s+3}, \end{aligned}$$

as required. \square

Pell numbers \mathcal{P}_n are defined recursively as $\mathcal{P}_1 = 1$, $\mathcal{P}_2 = 2$, and $\mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2}$ for all $n \geq 3$. The Binet's formula corresponding to \mathcal{P}_n is $\mathcal{P}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

Proposition 4.5. *Inside the $3 \times n$ table we have*

$$\mathcal{D}(s, 1) = \mathcal{P}_s \quad \text{and} \quad \mathcal{D}(s, 2) = \mathcal{Q}_s$$

for all $s \geq 1$. As a result,

$$\mathcal{D}(s, 1) \times \mathcal{D}(s+t, 2) - \mathcal{D}(s, 2) \times \mathcal{D}(s+t, 1) = (-1)^{s+1} \mathcal{D}(t, 1).$$

for all $s, t \geq 1$.

Proof. From the table in Lemma 3.2, we observe that

$$\begin{aligned} \mathcal{D}(s, 1) &= 2\mathcal{D}(s-1, 1) + \mathcal{D}(s-2, 1), \\ \mathcal{D}(s, 2) &= 2\mathcal{D}(s-1, 2) + \mathcal{D}(s-2, 2) \end{aligned}$$

for all $s \geq 3$. Now since $\mathcal{D}(1, 1) = \mathcal{P}_1 = 1$, $\mathcal{D}(2, 1) = \mathcal{P}_2 = 2$, $\mathcal{D}(1, 2) = \mathcal{Q}_1 = 1$, and $\mathcal{D}(2, 2) = \mathcal{Q}_2 = 3$ one can show, by using induction, that $\mathcal{D}(s, 1) = \mathcal{P}_s$ and $\mathcal{D}(s, 2) = \mathcal{Q}_s$ for all s . To prove the second claim, we use the following formula

$$\mathcal{P}_s \mathcal{Q}_{s+t} - \mathcal{Q}_s \mathcal{P}_{s+t} = (-1)^{s+1} \mathcal{P}_t$$

that can be proved simply using Binet's formulas. \square

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DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD,
P. O. BOX 1159, MASHHAD 91775, IRAN
E-mail address: daniel_yaqubi@yahoo.es